Exercise 3

1. Describe the following curves and express them in the form f(x, y) = 0.
   (a) $x = t^2 + 1,\ y = t - 1 \quad t \in \mathbb{R}$;
   (b) $x = \sin^2 t,\ y = \cos t, \quad t \in \mathbb{R}$;
   (c) $x = t \cos t,\ y = t \sin t, \quad t > 0$.

2. In plane geometry the ellipse is defined as the loci of the points whose sum of distance to two fixed points is constant. Let these two points be $(c, 0)$ and $(-c, 0), c > 0$ and $2a$ be the sum. Show the loci (x, y) satisfy the equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 , \quad b^2 = a^2 - c^2 .$$

3. * Study the cycloid
$$x = r(\alpha - \sin \alpha), \quad y = r(1 - \cos \alpha), \quad \alpha \in (-\infty, \infty),\ r > 0 .$$

4. Let $(c, 0)$ and $(-c, 0)$ be given and let H be the set of all points (x, y) whose difference in distances to $(c, 0)$ and $(-c, 0)$ is a constant $2a$.
   (a) Show that H is the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$
   (b) Show that it admits the parametric equations
$$x = \pm a \cosh t, \quad y = b \sinh t, \quad t \in \mathbb{R} .$$

5. Consider the Lissajous curve
$$x = A \sin(at + \delta), \quad y = B \sin bt , \quad t \in \mathbb{R} ,$$
where A, B, a, b, δ are positive constants. Show that the curve is closed if and only if a/b is a rational number. Here a curve is closed if there exists some T such that $(x(t), y(t)) = (x(t + T), y(t + T))$ for all t.

6. The folium of Descartes in parametric form is given by
$$x = \frac{3at}{1 + t^3}, \quad y = \frac{3at^2}{1 + t^3}, \quad a > 0 .$$
   (a) Show that it defines a regular curve on $(-\infty, -1)$ and $(-1, \infty)$.
   (b) Verify that it is the solution set to
$$x^3 + y^3 = 3axy .$$
   (c) Sketch its graph.

7. Find the velocity, speed and acceleration vectors of the following motions:
   (a) $\mathbf{x}(t) = (t^2, t^3)$,

(b) $\mathbf{x}(t) = (\cos t, \sin t, e^t)$,

(c) $\mathbf{r}(t) = (t, 4\tan t, 6t^2 - t^3)$.

8. Find the position $\mathbf{x}(t)$ of the motion in space when the acceleration and initial data are specified:

   (a)
   $$\mathbf{a}(t) = (6t, -1, 12t^2); \quad \mathbf{x}(0) = (0,0,0), \quad \mathbf{v}(0) = (1,1,0)\ .$$

   (b)
   $$\mathbf{a}(t) = (\cos t, \sin t, 1); \quad \mathbf{x}(0) = (100,20,0), \quad \mathbf{v}(0) = (0,0,5)\ .$$

9. A particle moves on the unit sphere centered at the origin with constant speed. Show that its velocity is always tangent to the sphere.

10. A particle moves along a parametric curve with constant speed. Prove that its acceleration is always perpendicular to its velocity.

11. Determine the maximum height of a projectile in the plane under the following information: Initial position (0,0), initial speed 160 m/sec, angle of inclination $\pi/3$.

12. A projectile is fired horizontally from a 1 km-cliff to reach 2 km from the base of the cliff. What should be the initial velocity ?

13. Let $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ be two differentiable curves from some interval to $\mathbb{R}^n$. Establish the following product rules:

   (a)
   $$\frac{d}{dt}\boldsymbol{\gamma}\cdot\boldsymbol{\eta} = \boldsymbol{\gamma}'\cdot\boldsymbol{\eta} + \boldsymbol{\gamma}\cdot\boldsymbol{\eta}'\ .$$

   (b)
   $$\frac{d}{dt}\boldsymbol{\gamma}\times\boldsymbol{\eta} = \boldsymbol{\gamma}'\times\boldsymbol{\eta} + \boldsymbol{\gamma}\times\boldsymbol{\eta}',$$

   when $n = 3$.

14. Let $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ be two regular curves on some interval and $\boldsymbol{\gamma}(t_1) = \boldsymbol{\eta}(t_2)$. Define the angle between these two curves at this point of intersection to the angle $\theta \in [0, \pi/2]$ between the two tangent lines passing this point. Show that

   $$\cos\theta = \frac{|\gamma_1'(t_1)\eta_1'(t_2) + \gamma_2'(t_1)\boldsymbol{\eta}'(t_2)|}{\sqrt{(\gamma_1'^2(t_1) + \eta_1'^2(t_2))(\gamma_2'^2(t_1) + \eta_2'^2(t_2))}}\ .$$

15. * Let γ and η be two regular curves in $\mathbb{R}^n$. Suppose $P = \boldsymbol{\gamma}(t_0)$ and $Q = \boldsymbol{\eta}(s_0)$ are points realizing the (minimal) distance between these two curves. Show that

   $$\overline{PQ}\cdot\boldsymbol{\gamma}'(t_0) = \overline{PQ}\cdot\boldsymbol{\eta}'(s_0) = 0.$$

16. The circle $x^2 + y^2 = 1$ can be described by the graphs of two functions, $f_1(x) = \sqrt{1-x^2}$ and $f_2(x) = 1\sqrt{1-x^2}$ over $[-1,1]$. However, both functions are not differentiable at $x = \pm 1$. Can you described the circle in terms of four differentiable functions over some intervals of the x- or y-axis ?

17. Write down the polar equation for a straight line not passing through the origin.

18. Show that the polar equation for the circle centered at $(a, 0)$ with radius a, where $a > 0$, is given by
$$\rho(\theta) = 2a\cos\theta, \quad \theta \in (-\pi/2, \pi/2]\ .$$

19. Sketch the graphs of the following polar equations and convert them to the form $f(x, y) = 0$.

   (a) The 3-leave Rhodonea
$$\rho = a\cos 3\theta\ , \quad a > 0.$$

   (b) The astroid
$$x = 4a\cos^3 t, \quad y = 4a\sin^3 t, \quad a > 0\ .$$

   (c) The logarithmic spiral
$$r = e^{b\theta}\ , \quad b > 0\ .$$

20. Consider the one-dimensional motion described by
$$m\frac{d^2x}{dt^2} = -kx\ ,$$
where $k > 0$ and $x(t)$ is the location of the particle at time t.

   (a) Show that $x'^2 + \frac{k}{m}x^2$ is constant in time.

   (b) Using (a) to show the motion must be of the form
$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}(t - t_0)\right), \quad \text{for some } A, t_0 \in \mathbb{R}\ .$$

   (c) Show that the general solution of the differential equation
$$m\frac{d^2x}{dt^2} = -kx + b\ , \quad b \in \mathbb{R},$$
   is given by
$$x(t) = \frac{b}{k} + A\cos\left(\sqrt{\frac{k}{m}}(t - t_0)\right).$$

21. * Find the length of the following parametric curves:

   (a)
$$\mathbf{r}(t) = (3\sin 2t, 3\cos 2t, 8t), \quad t \in [0, \pi].$$

   (b)
$$\mathbf{x}(t) = \left(t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{\sqrt{3}}\right), \quad t \in [0, 1]\ .$$

   (c)
$$\boldsymbol{\gamma}(t) = (2e^t, e^{-t}, 2t), \quad t \in [0, 1]\ .$$